

Towards p -Adic String in Constant B -Field

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Spacetime properties of the tachyon of the p -adic string theory can be derived from a (non-local) action on the p -adic line \mathbf{Q}_p , thought of as the boundary of the ‘worldsheet’. We show that a term corresponding to the background of the antisymmetric second rank tensor field B can be added to this action. We examine the consequences of this term, in particular, its relation to a noncommutative deformation of the effective theory of the p -adic tachyon in spacetime.

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1. Introduction

The tachyon sector of the D-branes of the open p -adic string theory is amenable to exact analytic calculation. It turns out that in many ways these theories behave like the ordinary bosonic string. The history of the p -adic string is also not unlike the ordinary string theory. It was introduced in Ref.[1], which proposed the Koba-Nielsen formula for the tree amplitude of N tachyons of the p -adic theory by generalizing the formula from the real numbers to the p -adic numbers. Namely, the integral was taken over (a quadratic extension of) the p -adic number field \mathbf{Q}_p and the absolute values in the integrands were replaced by the non-archimedean norm. It was shown that this prescription defines a consistent string theory. Soon afterwards it was realized[2-5] that these integrals can be computed exactly. The spacetime effective field theory of the tachyons is, therefore, known *exactly*. This was written down in Refs.[3,4]. (Let us notice that in this approach, spacetime is the usual Minkowski one. A more exotic type of p -adic string living in p -adic spacetime was studied in [6], but we will not consider it here.)

Although the prescription to define the p -adic string was well motivated mathematically and paid rich dividends, it did not shed much light on the nature of the p -adic string itself. The development in this direction came from the works of [7-11]. In Refs.[7-9], a non-local action on the p -adic field \mathbf{Q}_p was proposed for $X^\mu(\xi)$ ($\xi \in \mathbf{Q}_p$), the target space coordinates. Finally, a ‘worldsheet’ theory was given in [10,11]. The ‘bulk’ of the open string worldsheet turns out to be an infinite Bethe lattice, also called a Bruhat-Tits tree, with coordination number $p+1$; the boundary of which is isomorphic to \mathbf{Q}_p . The Polyakov action on the ‘worldsheet’ is the usual lattice action for the free massless fields X^μ . It was shown in [10] that starting with a finite lattice and inserting the tachyon vertex operators on the boundary, one recovers the prescription of [1-5] in the thermodynamic limit. For related works on the p -adic string theory, see [12-14], as well as the review [15].

More recently, the p -adic string theory have come to focus through the realization that the exact spacetime theory of its tachyon allows one to study nonperturbative aspects of string dynamics, like the process of tachyon condensation. In [16], the solitons of the effective theory of the p -adic tachyon[3,4,12] were identified with the D-branes and shown that the dynamics is according to the behaviour conjectured by Sen[17]. Moreover, it turns out that in the $p \rightarrow 1$ limit²[18], the theory provides an approximation to the boundary string field theory (BSFT)[19,20] of ordinary strings, the formalism of which was useful in

² See also the prescient comments in [7].

proving the Sen conjectures[18,21,22]. Various properties of the p -adic string have been explored recently in [23-30].

Despite these progress, much remains to be understood about the p -adic strings. We know some properties of its D-branes in flat space, though restricted mostly to the tachyon sector. Recently a first step towards understanding the behaviour of p -adic strings in a non-trivial background was taken in [31,32]. The motivation comes from the fact that a constant value of the second rank antisymmetric tensor field B in ordinary string theory has the effect of providing a noncommutative deformation of the open string fields in the target space. Thus the effective spacetime theory of the p -adic tachyon was deformed by introducing a noncommutative parameter θ . Gaussian solitons corresponding to D-branes were obtained for all values of θ and shown to interpolate smoothly from the p -adic soliton[3] to the noncommutative GMS soliton[33]. It was shown that this continues to be the case down to $p \rightarrow 1$, *i.e.*, in the BSFT of the ordinary string theory; thus providing a new insight to the effectiveness of the approach of Refs.[34] in understanding tachyon dynamics.

In this paper, we would like to examine a possible ‘worldsheet’ origin of the non-commutativity introduced in [31,32] to the effective action of the p -adic tachyon. We will modify the nonlocal action[7-9] on the *boundary* of the ‘worldsheet’ in analogy with the ordinary string theory in a constant B -field³. We show that the correlation functions with any number of the tachyon vertex operators can be solved in the presence of the coupling to the B -field (Sec. 3). Unfortunately, however, the deformation does not respect the complete $GL(2, \mathbf{Q}_p)$ symmetry—it is only invariant under the infinitesimal transformation—the underlying reason being the lack of a natural order in \mathbf{Q}_p . As a result, the four-tachyon scattering amplitude does not quite agree with that computed from the (deformed) effective field theory (Sec. 4). However, we discuss some formal ‘time ordered’ Green functions and related commutators of the tachyon field. As an alternative approach, we explore the possibilities of deforming the Koba-Nielsen amplitudes (Sec. 5) and end with some comments (Sec. 6). For completeness, the ‘worldsheet’ approach to the p -adic string, developed in Refs.[10] and [7-9], in flat spacetime with no background field, is briefly recapitulated

³ It turns out that this can be done consistently for a subset of primes p , namely $p = 3 \pmod{4}$, which roughly are half of all the prime numbers. However, once we are in the domain of the effective theory in spacetime, there is no reason why the results cannot be continued beyond this set, indeed to all integers.

(Sec. 1). Two appendices provide a collection of results from p -adic analysis useful for our purpose (Appendix A); and a brief derivation of an infrared regulated integral (Appendix B).

2. ‘Worldsheet’ action of the p -adic string in the trivial background

In this section we will review the ‘worldsheet’ approach to the p -adic open string in flat spacetime with no other background field, as proposed in Refs.[7-11].

Let us start by recalling the ‘worldsheet’ construction of the p -adic string given in Ref.[10]. The interior of the worldsheet, analogous to the unit disc or the upper half-plane of the usual theory is an infinite lattice with no closed loops, *i.e.*, a tree \mathcal{T}_p in which $p + 1$ edges meet at each vertex (see Fig. 1). This is known to mathematicians as the Bruhat-Tits tree and is familiar to physicists as the Bethe lattice. The boundary of the tree \mathcal{T}_p , defined as the union of all infinitely remote vertices, can be identified with the p -adic field \mathbf{Q}_p . On the other hand, the tree \mathcal{T}_p is the (discrete) homogeneous space $\mathrm{PGL}(2, \mathbf{Q}_p) / \mathrm{PGL}(2, \mathbf{Z}_p)$: the coset obtained by modding $\mathrm{PGL}(2, \mathbf{Q}_p)$ by its maximal compact subgroup⁴ $\mathrm{PGL}(2, \mathbf{Z}_p)$. The action of $\mathrm{PGL}(2, \mathbf{Q}_p)$ on \mathbf{Q}_p extends naturally to \mathcal{T}_p .

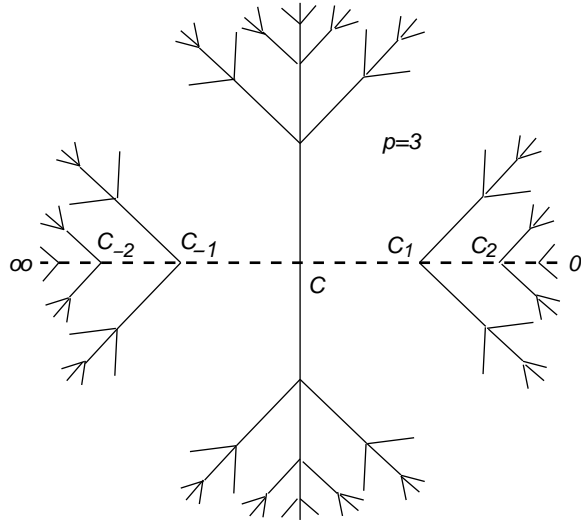


Fig. 1: A finite part of the ‘worldsheet’ of the 3-adic string: the tree \mathcal{T}_p for $p = 3$.

⁴ This construction parallels the case of the usual string theory, in which the UHP is the homogeneous coset $\mathrm{PSL}(2, \mathbf{R})$ modulo its maximal compact subgroup $\mathrm{SO}(2)$.

Let z label the vertices of \mathcal{T}_p and e its edges. We will use ξ to label the points on the boundary $\partial\mathcal{T}_p \equiv \mathbf{Q}_p$. The spacetime coordinates $X^\mu(z)$ are functions on the lattice. The difference $\Delta_e X^\mu = X^\mu(z') - X^\mu(z)$, where z and z' are the end-points of the edge e , is the finite lattice analogue of the directional derivative. Likewise, the Polyakov action on the p -adic worldsheet is the free action

$$S_p[X] = \frac{1}{2}\beta_p \sum_{\{e\}} (\Delta_e X^\mu(z))^2. \quad (1)$$

The variation of the above gives rise to

$$\delta S_p = -\beta_p \left[\sum_{\{z\}} \delta X^\mu(z) \nabla^2 X^\mu(z) + (p-1) \int_{\partial\mathcal{T}_p} d\mu_\xi \delta X^\mu(\xi) \mathcal{D}_n^{(p)} X^\mu(\xi) \right], \quad (2)$$

where

$$\nabla^2 X^\mu(z) = \sum_{i=1}^{p+1} X^\mu(z_i) - (p+1)X^\mu(z), \quad (3)$$

(z_i are the $p+1$ nearest neighbours of z), is the lattice Laplacian, $\mathcal{D}_n^{(p)} X^\mu(\xi)$ is an appropriately defined normal derivative⁵ at the boundary point ξ and $d\mu_\xi$ is a measure on the boundary \mathcal{T}_p (see [10] for the precise definitions). The tension of the p -adic string is β_p . We see that either the Neumann ($\mathcal{D}_n^{(p)} X^\nu(\xi) = 0$) or the Dirichlet ($X^\nu(\xi) = \text{constant}$) boundary condition can be imposed on a spacetime coordinate X^ν . The solutions of the classical equations of motion $\nabla^2 X^\mu(z) = 0$ are harmonic functions on the tree \mathcal{T}_p .

We are interested in the scattering amplitude of N tachyonic scalars. This requires us to compute the correlation function of the N vertex operators $\exp(i k^I \cdot X)$, $(k^I)^2 = 2$, corresponding to the tachyon with momentum k^I ($I = 1, \dots, N$), inserted at the boundary points $\xi_I \in \partial\mathcal{T}_p = \mathbf{Q}_p$. To this end, we work with a finite part of the lattice of radius R centred at the point C (see Fig. 1) and define the scattering amplitude $\mathcal{A}_p^{(N)}$ as the limit $R \rightarrow \infty$ of the finite lattice correlator:

$$\mathcal{A}_p^{(N)} = \lim_{R \rightarrow \infty} \frac{1}{Z_p} \int \mathcal{D}X \exp \left(-S_p[X] + i \sum_{I=1}^N k^I \cdot X(\xi_I) \right), \quad (4)$$

⁵ This is defined in terms of the limit of difference of $f(z)$ and $f(\xi)$ as the interior point z of the tree \mathcal{T}_p approach the boundary $\mathbf{Q}_p = \partial\mathcal{T}_p$.

where ξ_I are points on the boundary at radius R . The measure $\mathcal{D}X = \prod_{\mu,z} dX^\mu(z)$ is the usual measure and the normalization in (4) is by the partition function

$$Z_p = \int \mathcal{D}X e^{-S_p[X]}. \quad (5)$$

The regularity of the thermodynamic limit $R \rightarrow \infty$ determines the string tension to be $\beta_p = 1/\ln p$, which also happens to be the condition for the projective $\text{GL}(2, \mathbf{Q}_p)$ invariance of the tachyon scattering amplitude. In Ref.[10], it is shown that the expression (4) is the p -adic analogue of the Koba-Nielsen amplitude

$$\mathcal{A}_p^{(N)} = \int_{\mathbf{Q}_p} \prod_{I=1}^{N-3} d\xi_I |\xi_I|_p^{k^I \cdot k^N} |1 - \xi_I|_p^{k^I \cdot k^{N-1}} \prod_{1 \leq I < J \leq N-3} |\xi_I - \xi_J|_p^{k^I \cdot k^J}, \quad (6)$$

where the projective invariance is used to fix the boundary points $\xi_{N-2} = 0$, $\xi_{N-1} = 1$ and $\xi_N = \infty$.

It is also possible to integrate over the degrees of freedom in the interior of the tree \mathcal{T}_p to arrive at a non-local action on the boundary (see Appendix C of Ref.[10] for details):

$$\mathcal{S}_p = \frac{p(p-1)\beta_p}{4(p+1)} \int_{\mathbf{Q}_p} d\xi d\xi' \frac{(X^\mu(\xi) - X^\mu(\xi'))^2}{|\xi - \xi'|_p^2}. \quad (7)$$

This action was proposed in [7-9] and shown to yield the correlators of the tachyon vertex operators inserted on the boundary. In the following, this will be our starting point for coupling the B -field. Let us note that in the case of the usual bosonic string theory too, there is an analogue of this non-local action—it is the one obtained by the obvious substitutions in Eq.(7).

3. A constant B -field background in the boundary effective action

In ordinary string theory, a constant background B -field contributes a total derivative in the worldsheet action. Hence its only effect is in the boundary

$$\frac{1}{2} \int_{\Sigma} d^2 z \varepsilon^{\alpha\beta} \partial_\alpha X^\mu(z) \partial_\beta X^\nu(z) B_{\mu\nu} = \int_{\partial\Sigma} d\xi B_{\mu\nu} X^\mu(\xi) \partial_t X^\nu(\xi), \quad (8)$$

where ∂_t is the tangential derivative along the boundary $\partial\Sigma$, which we label by ξ . (Formally, this is the same as the insertion of the vertex operator for a background gauge field $A_\mu[X(\tau)] \sim B_{\mu\nu} X^\nu(\tau)$.) In principle therefore, it is a straightforward problem to generalize the effect of a constant B -field in the p -adic string theory: add the p -adic analogue

of the boundary term (8) to the action (7). However, one is immediately faced with a problem: there is no natural notion of a (tangential) derivative along the p -adic line \mathbf{Q}_p (see [35-38] for aspects of p -adic analysis).

The same problem was encountered by the authors of [13] in their computation of the correlation function involving the vertex operator of a vector field $\epsilon_\mu(k) \partial_t X^\mu e^{ik \cdot X}(\xi)$ in the p -adic string theory. The solution proposed there is to use the Cauchy-Riemann relations for a pair of harmonic functions in the usual case, and write the tangential derivative as [13,35]

$$\partial_t^{(p)} X^\mu(\xi) \equiv \partial_\xi X^\mu = \int_{\mathbf{Q}_p} d\xi' \frac{\text{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} X^\mu(\xi'), \quad (9)$$

where $\text{sgn}_\tau(\xi)$ is an analogue of the sign function over \mathbf{R} . More precisely, it is a so called *multiplicative branching character* on \mathbf{Q}_p corresponding to the quadratic extension⁶ $\mathbf{Q}_p(\sqrt{\tau})$ by $\sqrt{\tau}$. It turns out that, up to equivalence, there are three choices $\tau = \varepsilon, p$ and εp (where ε is a $(p-1)$ -th root of unity). However, if we demand the antisymmetry property of the sign function, as in \mathbf{R} :

$$\text{sgn}_\tau(-\xi) = -\text{sgn}_\tau(\xi). \quad (10)$$

holds only for the last two choices (namely, $\tau = p, \varepsilon p$) with an *additional restriction* on the value of the prime $p = 3 \pmod{4}$, which is satisfied by roughly half the prime numbers [35]. The antisymmetry property (10) is of importance to us, so we will restrict to these values of p . For a $\xi \in \mathbf{Q}_p$, $\text{sgn}_\tau(\xi)$ is defined as follows:

$$\text{sgn}_\tau(\xi) = \begin{cases} +1, & \text{if } \xi = \zeta_1^2 - \tau \zeta_2^2 \text{ for some } \zeta_1, \zeta_2 \in \mathbf{Q}_p, \\ -1 & \text{otherwise.} \end{cases} \quad (11)$$

Let us recall that the function sgn_τ was used in defining p -adic string amplitudes with Chan-Paton factors [3,15,39].

We are now in a position to put forward our proposal for the boundary effective action in the presence of a constant background B -field. It is:

$$\begin{aligned} \mathcal{S}_p(B) = & \frac{T_0}{2} \left[\int_{\mathbf{Q}_p} \eta_{\mu\nu} \frac{(X^\mu(\xi) - X^\mu(\xi'))(X^\nu(\xi) - X^\nu(\xi'))}{|\xi - \xi'|_p^2} d\xi d\xi' \right. \\ & \left. + i \frac{1+p}{p^2 \Gamma_\tau(-1)} \int_{\mathbf{Q}_p} B_{\mu\nu} X^\mu(\xi) \frac{\text{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} X^\nu(\xi') d\xi d\xi' \right], \end{aligned} \quad (12)$$

⁶ The subscript $\tau \in \mathbf{Q}_p$ refers to a p -adic number such that $\sqrt{\tau} \notin \mathbf{Q}_p$. One obtains a quadratic extension $\mathbf{Q}_p(\sqrt{\tau})$ of the p -adic field by adjoining $\sqrt{\tau}$; *i.e.*, by considering elements of the form $\xi + \sqrt{\tau}\eta$ for $\xi, \eta \in \mathbf{Q}_p$. Unlike \mathbf{R} the quadratic extension of \mathbf{Q}_p is not unique. In other words, there are several choices for τ and the function sgn_τ depends on this choice. It is worth emphasizing, however, that sgn_τ is a (real valued) function on \mathbf{Q}_p , and does not require that we extend it.

where $\Gamma_\tau(\xi)$ is a generalized gamma function over the p -adic field⁷ discussed in Appendix A and

$$T_0 = \frac{p(p-1)}{2(p+1)\ln p} \frac{1}{\alpha'} \quad (13)$$

is the ‘string tension’. The antisymmetry of the B -field requires that at least two directions are involved, thus the rank of B is always even. In the following, although we will not be precise about it: if a (spatial) coordinate X^μ is involved, it is assumed that B has a component in that direction.

The normal derivative[7-10] on \mathbf{Q}_p

$$\partial_n^{(p)} f(\xi) = \int_{\mathbf{Q}_p} d\xi' \frac{f(\xi') - f(\xi)}{|\xi' - \xi|_p^2} \quad (14)$$

appears in the first line of Eq.(12). Let us note in parenthesis that a normal derivative $\mathcal{D}_n^{(p)}$ was introduced in Eq.(2). Neither the Haar measure $d\xi$ nor the normal derivative (14), match the corresponding quantities in Eq.(2). However, the combination $d\xi \partial_n^{(p)} f(\xi)$ equals $d\mu_\xi \mathcal{D}_n^{(p)} f$ (see [10] for details).

3.1. Correlators in the presence of B -field

In the rest of the section, we will analyze various consequences of action (12) using standard field theoretic techniques. We recall that the N -point correlation function of the tachyon vertex operator $e^{ik \cdot X}(\xi)$ in a constant B -field background is given by the correlator

$$\langle e^{ik_1 \cdot X}(\xi_1) \dots e^{ik_N \cdot X}(\xi_N) \rangle_B = \frac{\int \mathcal{D}X \exp \left(-\mathcal{S}_p(B) + i \sum_{I=1}^N k^I \cdot X(\xi_I) \right)}{\int \mathcal{D}X \exp(-\mathcal{S}_p(B))}. \quad (15)$$

Let us introduce the generating function

$$\mathcal{Z}[J] = \int \mathcal{D}X \exp \left[-\mathcal{S}_p(B) + i \int_{\mathbf{Q}_p} d\xi J \cdot X(\xi) \right], \quad (16)$$

which, when the external current $J_\mu(\xi) = \sum_{I=1}^N k_\mu^I \delta(\xi - \xi_I)$, yields the N -point correlation function (15). The generating function $\mathcal{Z}[J]$ satisfies the Dyson-Schwinger equation

$$\int_{\mathbf{Q}_p} d\xi' \Delta_{\mu\nu}(\xi - \xi') \frac{\delta \ln \mathcal{Z}[J]}{\delta J_\nu(\xi')} = -J_\mu(\xi), \quad (17)$$

⁷ The generalized gamma functions $\Gamma_\tau(s)$ associated with sgn_τ , is not singular at zero or negative integer arguments, or indeed anywhere on the complex s -plane.

where the operator $\Delta_{\mu\nu}$

$$\Delta_{\mu\nu} = T_0 \left[-\eta_{\mu\nu} \partial_n^{(p)} + i \frac{1+p}{p^2 \Gamma_\tau(-1)} B_{\mu\nu} \partial_t^{(p)} \right] \quad (18)$$

is a combination of the normal derivative $\partial_n^{(p)}$ and tangential derivative $\partial_t^{(p)}$ defined in Eqs.(14) and (9) respectively.

It remains to solve the Green function $\mathcal{G}^{\mu\nu}(\xi - \xi')$ which is a kernel to the differential operator (18)

$$\int_{\mathbf{Q}_p} d\xi'' \Delta_{\mu\lambda}(\xi - \xi'') \mathcal{G}^{\lambda\nu}(\xi'' - \xi') = \delta_\mu^\nu \delta(\xi - \xi'). \quad (19)$$

To this end, we solve the ‘equation of motion’:

$$\int_{\mathbf{Q}_p} d\xi' \Delta_{\mu\nu}(\xi - \xi') X^\nu(\xi') = -J_\mu(\xi), \quad (20)$$

by taking a Fourier transformation on the p -adic number \mathbf{Q}_p , as we do for ordinary strings (See Appendix A and [35-38] for details). In the Fourier space Eq.(20) takes the form

$$T_0 \frac{1+p}{p^2} (\eta_{\mu\nu} - i B_{\mu\nu} \text{sgn}_\tau(\omega)) |\omega|_p \tilde{X}^\nu(\omega) = -\tilde{J}_\mu(\omega). \quad (21)$$

Introducing the open string metric $G_{\mu\nu}$ and the theta parameter $\theta^{\mu\nu}$ by the relation

$$G^{-1} + \frac{i}{2} \frac{p-1}{\alpha' p \ln p \Gamma_\tau(0)} \theta = \left(\frac{1}{\eta - iB} \right), \quad (22)$$

we find that

$$\tilde{X}^\mu(\omega) = -\frac{p^2}{T_0(1+p)} \left(G^{\mu\nu} + \frac{i}{2} \frac{p-1}{\alpha' p \ln p \Gamma_\tau(0)} \theta^{\mu\nu} \text{sgn}_\tau(\omega) \right) \frac{1}{|\omega|_p} \sum_{I=1}^N k_\nu^I \chi_p(-\omega \xi_I), \quad (23)$$

where $\chi_p(\omega)$ is the p -adic analogue of the function $e^{i\omega}$ (see Appendix A). We have also used the expression of the external current $J_\mu = \sum_{I=1}^N k_\mu^I \delta(\xi - \xi_I)$, appropriate for (15). Finally, after an inverse Fourier transform, we arrive at the desired expression

$$X^\mu(\xi) = \sum_{I=1}^N k_\nu^I \left(\alpha' G^{\mu\nu} \ln |\xi - \xi_I|_p - \frac{i}{2} \theta^{\mu\nu} \text{sgn}_\tau(\xi - \xi_I) \right), \quad (24)$$

To be precise, the expression above is obtained after an ‘infrared regularization’. It is similar to the usual string theory and the details are given in Appendix B.

The Green's function is therefore

$$\mathcal{G}^{\mu\nu}(\xi - \xi') = -\alpha' G^{\mu\nu} \ln |\xi - \xi'|_p + \frac{i}{2} \theta^{\mu\nu} \text{sgn}_\tau(\xi - \xi'), \quad (25)$$

in terms of which we have a complete solution

$$\mathcal{Z}[J] = \exp \left(-\frac{1}{2} \sum_{I,J=1}^N k_\mu^I k_\nu^J \mathcal{G}^{\mu\nu}(\xi_I - \xi_J) \right) \quad (26)$$

up to an irrelevant constant of integration. In particular, the N -point correlation function (15) is:

$$\left\langle e^{ik^1 \cdot X}(\xi_1) \cdots e^{ik^N \cdot X}(\xi_N) \right\rangle_B = \prod_{\substack{I,J=1 \\ I < J}}^N \frac{\exp \left(-\frac{i}{2} \theta^{\mu\nu} k_\mu^I k_\nu^J \text{sgn}_\tau(\xi_I - \xi_J) \right)}{|\xi_I - \xi_J|_p^{-\alpha' G^{\mu\nu} k_\mu^I k_\nu^J}}, \quad (27)$$

where we have omitted the momentum conserving delta function. This is the p -adic analogue of the result in Ref.[40] for the usual bosonic string and has the same form as the latter.

3.2. Formal consequences of the B -field

Notwithstanding its apparently nice expression and the similarity with the real case, there are problems with (27). It is not invariant under projective invariance of \mathbf{Q}_p , *i.e.*, under $\text{GL}(2, \mathbf{Q}_p)$ transformations. Looking back, we find that this is indeed a problem with (12). As a matter of fact, even in the real case, the amplitudes are invariant under those $\text{SL}(2, \mathbf{R})$ transformations which preserve the cyclic ordering of the tachyon vertex operators. It is here that we face a fundamental problem: one cannot define an order in \mathbf{Q}_p that is compatible with its algebraic properties. The function sgn_τ is therefore not associated with any ordering. The best we can do is to prove invariance under infinitesimal $\text{SL}(2, \mathbf{Q}_p)$ transformation. However, before we sketch the proof, let us provide some formal argument, which show that the B -field background gives rise to spacetime noncommutativity, at least formally.

We start by defining an ordering ('time ordering') in \mathbf{Q}_p . This can be done by ordering the coefficients in the power series expansion of a p -adic number: $\xi = p^N (\xi_0 + \xi_1 p + \cdots)$ (given in Appendix A), but let us add a further ingredient to it. The first coefficient ξ_0 can be any of the non-zero elements of the finite field $\mathbf{F}_p = \mathbf{Z}_p / p\mathbf{Z}_p \simeq \mathbf{Z} / p\mathbf{Z}$. Moreover, a

primitive $(p-1)$ -th root of unity exists[35] in \mathbf{F}_p . In other words, there exists $\eta \in \mathbf{F}_p^*$, such that $\eta^{p-1} = 1$ and powers of η generates \mathbf{F}_p^* . This implies that out of the $(p-1)$ values of ξ_0 , exactly half are odd/even powers of η . It is now possible to show that for a fixed N , *i.e.*, for $|\xi|_p = p^{-N}$, exactly half have $\text{sgn}_\tau(\xi) = +1$ and the other half have $\text{sgn}_\tau(\xi) = -1$. In terms of the ‘worldsheet’ in Fig. 1, at each node along the dashed line from zero to infinity, $(p-1)$ branches labelled by ξ_0 originate. Of these, $(p-1)/2$ correspond to a p -adic number ξ with $\text{sgn}_\tau(\xi) = +1$ (respectively -1). By convention, we can say that the positive (negative) set is the other above (below) the dashed line, and each half can be ordered by the norm and the coefficients in the power series. The purpose of this exercise is to show that we can approach $\xi = 0$ through a sequence of points ξ_I such that $|\xi_I|_p \rightarrow 0$ and $\text{sgn}_\tau(\xi_I)$ fixed to be either of ± 1 , exactly as in the case of the real string.

We can now follow standard procedure[41,42] to write

$$\begin{aligned}
[X^\mu(0), X^\nu(0)] &= T(\langle X^\mu(0)X^\nu(0-) \rangle - \langle X^\mu(0)X^\nu(0+) \rangle) \\
&= \lim_{\substack{\text{sgn}_\tau(\xi)=-1 \\ |\xi|_p \rightarrow 0}} \langle X^\mu(0)X^\nu(\xi) \rangle - \lim_{\substack{\text{sgn}_\tau(\xi)=+1 \\ |\xi|_p \rightarrow 0}} \langle X^\mu(0)X^\nu(\xi) \rangle \\
&= i\theta^{\mu\nu},
\end{aligned} \tag{28}$$

where, we have made use of the Green function (25). It is possible to rephrase it in another way. The exact expression for the correlators (27), can be written equivalently as the formal operator product expansion between the tachyon vertex operators on \mathbf{Q}_p :

$$:e^{ik \cdot X}(\xi) :: e^{ik' \cdot X}(\xi') : = \frac{\exp\left(-\frac{i}{2}k_\mu k'_\nu \theta^{\mu\nu} \text{sgn}_\tau(\xi - \xi')\right)}{|\xi - \xi'|_p^{-\alpha' G^{\mu\nu} k_\mu k'_\nu}} : e^{i(k+k') \cdot X}(\xi) : + \dots \tag{29}$$

Let us introduce, following Ref.[41],

$$\Phi(X(\xi)) = \frac{1}{(2\pi)^D} \int_{\mathbf{R}^D} d^D k \tilde{\Phi}(k) e^{ik \cdot X}(\xi), \tag{30}$$

where $\tilde{\Phi}(k)$ is the Fourier transform of the function $\Phi(x)$. Notice that the object (30) is similar to a string field. More precisely, in the usual string theory in which ξ labels the real line, $\phi(x) = \lim_{\xi \rightarrow 0} \Phi(X(\xi))|0\rangle$ is the string field for the tachyon. Remaining in the real case, we recall that the operator product expansion of the vertex operators is equivalent to the statement about multiplication of the objects in (30). Without a B -field, $\Phi(X(1))\Psi(X(0)) = (\Phi\Psi)(X(0)) + \dots$; this gets deformed to a noncommutative product $\Phi \star \Psi$ in the presence of a constant B -field.

Returning to the case of the p -adic string, we will evaluate the product $\Phi(X(1))\Psi(X(0))$. (Parenthetically, evaluating this with $\xi = 1$ and $\xi' = 0$ assumes that the $\text{GL}(2, \mathbf{Q}_p)$ invariance can be used to find a more general expression[43].)

$$\begin{aligned}\Phi(X(1))\Psi(X(0)) &= \frac{1}{(2\pi)^D} \int_{\mathbf{R}^D} d^D k (\widetilde{\Phi \star \Psi})(k) e^{ik \cdot X(0)}, \\ (\widetilde{\Phi \star \Psi})(k) &= \frac{1}{(2\pi)^D} \int_{\mathbf{R}^D} d^D k' \tilde{\Phi}(k') \tilde{\Psi}(k - k') e^{-\frac{i}{2} k_\mu k'_\nu \theta^{\mu\nu}}.\end{aligned}\tag{31}$$

Thus, it seems that as a result of the B -field in (12), ordinary pointwise multiplication of functions of spacetime is deformed to the noncommutative Moyal product.

It would, however, be prudent to interpret the above results with some caution. The arguments we have used are rather formal. The B -field does not respect $\text{GL}(2, \mathbf{Q}_p)$ invariance. And the order that we have defined is not compatible with the algebraic properties of the field \mathbf{Q}_p . In particular, it is not invariant under the projective symmetry. In order to demonstrate that the B -field leads to spacetime noncommutativity, one needs to calculate all the tachyon N -point functions, in the presence of the constant B -field. We will discuss the difficulties with this in the next section.

3.3. Infinitesimal projective invariance

Let us study the p -adic analog of the projective (Möbius) transformation $\text{GL}(2, \mathbf{Q}_p)$ on the correlation functions (27). The coordinates ξ_I ($I = 1, \dots, N$) of the vertex operators are transformed as

$$\xi_I \rightarrow \xi'_I = \frac{a\xi_I + b}{c\xi_I + d}, \quad ad - bc \neq 0,\tag{32}$$

where $a, b, c, d \in \mathbf{Q}_p$. It is easy to see that the integrated N -point correlation function

$$\mathcal{A}_p^{(N)}(k_1, \dots, k_N) = \int_{\mathbf{Q}_p} \prod_{i=1}^N d\xi_i \left\langle e^{ik^1 \cdot X(\xi_1)} \dots e^{ik^N \cdot X(\xi_N)} \right\rangle_B\tag{33}$$

is invariant if

$$\text{sgn}_\tau(\xi'_I - \xi'_J) = \text{sgn}_\tau(\xi_I - \xi_J),\tag{34}$$

is true. In the usual case of the real strings, this is true under $\text{SL}(2, \mathbf{R})$ transformations which preserve the cyclic ordering of $\{\xi_1, \xi_2, \dots, \xi_N\}$. In the p -adic case, the absence of an order compatible with the algebraic properties of \mathbf{Q}_p , prevents us from making an

analogous restriction. We will therefore have to be content with a limited invariance. Namely, we will prove Eq.(34) for an infinitesimal transformation:

$$\xi_I \rightarrow \xi'_I = \xi_I + \epsilon_{-1} + \epsilon_0 \xi_I + \epsilon_1 \xi_I^2, \quad (35)$$

which we require to satisfy

$$|\epsilon_0|_p < 1, \quad |\epsilon_1|_p \cdot |\xi_I|_p < 1, \quad (I = 1, 2, \dots, N). \quad (36)$$

Therefore, $\text{sgn}_\tau(\xi'_I - \xi'_J) = \text{sgn}_\tau(\xi_I - \xi_J) \text{sgn}_\tau(1 + \epsilon_0 + \epsilon_1(\xi_I + \xi_J))$. Thanks to the non-archimedian property: $|\xi_i + \xi_j|_p \leq \max(|\xi_i|_p, |\xi_j|_p)$ and the condition (36),

$$1 + \epsilon_0 + \epsilon_1(\xi_I + \xi_J) \in 1 + p\mathbf{Z}_p.$$

It can be shown that the sign function (11) of p -adic numbers of this form is positive[35]. This proves the invariance of the integrated correlation function (33) under infinitesimal projective transformation. Unfortunately, however, this is not sufficient to fix the positions of three of the vertex operators.

4. Tachyon scattering amplitudes

In this section, we evaluate the N tachyon scattering amplitudes, specifically for $N = 3, 4$. The lack of $\text{GL}(2, \mathbf{Q}_p)$ invariance due to the B -field does not, strictly speaking, allow us to fix three of the positions ξ_I . We will, nevertheless, fix ξ_1, ξ_2 and ξ_N by hand to 0, 1 and ∞ . The last one requires some care, as we can approach ‘infinity’ through a sequence of ‘positive’ or ‘negative’ p -adic numbers (see the discussion in Sec. 3.2). We can average over the two, however, it turns out that the final result is not sensitive to the details as long as the sequence of points have the same value of sgn_τ .

Let us consider the three-tachyon amplitude to begin with. Ignoring the momentum conserving delta function and an infinite factor, the result is $\mathcal{A}_p^{(3)}(k_1, k_2, k_3) = e^{\frac{i}{2}(k_1 \theta k_2)}$. Since there is no natural notion of a cyclic (or indeed any) ordering in \mathbf{Q}_p , we need to add by hand a term with the role of k_1 and k_2 interchanged and average over the two. The final result is:

$$\mathcal{A}_p^{(3)}(k_1, k_2, k_3) = \cos \frac{1}{2}(k_1 \theta k_2). \quad (37)$$

This expression is the same as that of the usual bosonic string.

Consider the case $N = 4$ next. Once again, fixing ξ_1 , ξ_2 and ξ_4 by hand, we arrive at the integral:

$$e^{\frac{i}{2}k_1\theta k_2} \int_{\mathbf{Q}_p} d\xi |\xi|_p^{k_1 \cdot k_3} |1 - \xi|_p^{k_2 \cdot k_3} \exp \left(\frac{i}{2}(k_1\theta k_3) \text{sgn}_\tau(\xi) - \frac{i}{2}(k_2\theta k_3) \text{sgn}_\tau(1 - \xi) \right). \quad (38)$$

This can be evaluated in two ways: either we can expand the exponential and use the expression of generalized p -adic beta-/gamma-functions (given in Appendix A), or we can divide the integral in four domains as in [3]. Following the first route, we get, after averaging with a term with $(k_1 \leftrightarrow k_2)$:

$$\begin{aligned} \mathcal{A}_p^{(4)} &= c_{12}c_{13}c_{23} \Gamma(\alpha(s)) \Gamma(\alpha(t)) \Gamma(\alpha(u)) + s_{12}s_{13}c_{23} \Gamma_\tau(\alpha(s)) \Gamma_\tau(\alpha(t)) \Gamma(\alpha(u)) \\ &\quad - s_{12}c_{13}s_{23} \Gamma_\tau(\alpha(s)) \Gamma(\alpha(t)) \Gamma_\tau(\alpha(u)) + c_{12}s_{13}s_{23} \Gamma(\alpha(s)) \Gamma_\tau(\alpha(t)) \Gamma_\tau(\alpha(u)) \\ &= c_{12}c_{34} \left[\frac{p-1}{p} \frac{1}{p^{\alpha(s)}-1} - \frac{1}{p} \right] + c_{13}c_{24} \left[\frac{p-1}{p} \frac{1}{p^{\alpha(t)}-1} - \frac{1}{p} \right] \\ &\quad + c_{14}c_{23} \left[\frac{p-1}{p} \frac{1}{p^{\alpha(u)}-1} - \frac{1}{p} \right] + c_{12}c_{13}c_{23} \frac{p+1}{p}, \end{aligned} \quad (39)$$

where, we have used the abbreviations $c_{IJ} = \cos \frac{1}{2}k_{I\mu}\theta^{\mu\nu}k_{J\nu}$ and $s_{IJ} = \sin \frac{1}{2}k_{I\mu}\theta^{\mu\nu}k_{J\nu}$ and as usual, $\alpha(s) = \frac{s}{2} = k_1 \cdot k_2 + 1$, etc. Alternatively, following [3], we divide \mathbf{Q}_p into \mathbf{Z}_p and its complement \mathcal{D}_1 ; then divide \mathbf{Z}_p further into three domains: \mathcal{D}_2 , in which the first coefficient ξ_0 in the expansion of ξ is 0; \mathcal{D}_3 corresponding to $\xi_0 = 1$ and \mathcal{D}_4 corresponding to the union of $\xi_0 = 2, 3, \dots, p-1$. In $\mathcal{D}_{1,2,3}$ the integral gives $\frac{p-1}{p} \frac{e^{\frac{i}{2}k_1\theta k_2} c_{34}}{p^{\alpha(s)}-1}$, $\frac{p-1}{p} \frac{e^{\frac{i}{2}k_2\theta k_4} c_{13}}{p^{\alpha(t)}-1}$ and $\frac{p-1}{p} \frac{e^{-\frac{i}{2}k_1\theta k_4} c_{23}}{p^{\alpha(u)}-1}$ respectively. In the region \mathcal{D}_4 , we are unable to do the calculation for an arbitrary prime p . However, explicit evaluation for the first few relevant primes leads to the result

$$\frac{e^{\frac{i}{2}(k_1\theta k_2)}}{p} \left[(p-3) c_{13}c_{23} + e^{-\frac{i}{2}(k_1\theta k_3) + \frac{i}{2}(k_2\theta k_3)} \right], \quad (40)$$

which, when averaged with the term with $(k_1 \leftrightarrow k_2)$, agrees with (39). Recall that in the commutative case, this region (and likewise its analogues for higher N) gives the four-tachyon (respectively N -tachyon) vertex in the spacetime effective field theory, as the integrand here is independent of the spacetime momenta in that case. With the B -field turned on, there is always momentum dependence, but in this region, it is only in the ‘phase’ factors. Let us also note that the presence of the Γ_τ -function in the four-tachyon amplitudes makes it impossible for these to have any adelic property.

Unfortunately, the four-point function is not symmetric under a permutation of the four momenta. Even if we consider summing up the terms with different permutations by

hand, the resulting expression does not match the results of the noncommutative deformation of the tachyon effective field theory considered in [32]. The latter is a field theory with the action:

$$\mathcal{L}_{NC}^p(T) = \frac{p}{p-1} \left[-\frac{1}{2} T \star p^{-\frac{1}{2}} \square^{-1} T + \sum_{N=3}^{p+1} \frac{(p-1)!}{N!(p-N+1)!} \left(\frac{g}{p} \right)^N (\star T)^N \right], \quad (41)$$

where T is the tachyon field on the D-brane and g is the open string coupling. The three- and four-tachyon amplitudes from this field theory are:

$$\begin{aligned} \mathcal{A}_{NC}^{p(3)} &\sim \frac{p-1}{p} c_{12} \\ \mathcal{A}_{NC}^{p(4)} &\sim \frac{p-1}{p} \left(\frac{c_{12}c_{34}}{p^{\alpha(s)}-1} + \frac{c_{13}c_{24}}{p^{\alpha(t)}-1} + \frac{c_{14}c_{23}}{p^{\alpha(u)}-1} \right) \\ &\quad + \frac{p-2}{p} (c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}). \end{aligned} \quad (42)$$

Finally, let us contrast the result (39) with the four-tachyon amplitude in the usual noncommutative bosonic string theory:

$$\begin{aligned} A_B^{(4)} &= \cos \frac{1}{2} (k_1 \theta k_3 - k_2 \theta k_4) B(\alpha(s), \alpha(t)) + \cos \frac{1}{2} (k_1 \theta k_4 + k_2 \theta k_3) B(\alpha(t), \alpha(u)) \\ &\quad + \cos \frac{1}{2} (k_1 \theta k_2 + k_3 \theta k_4) B(\alpha(u), \alpha(s)), \end{aligned} \quad (43)$$

where we have averaged over cyclic and anti-cyclic permutations and $B(\alpha, \beta)$ is the beta function.

5. Alternative proposal for the four-tachyon amplitude

In this section, we will explore the possibility of defining four-tachyon amplitudes of p -adic string in a constant B -field so as to reproduce the noncommutative deformation (41) of the effective field theory[32]. To this end, let us recall the situation for the usual bosonic string theory. Either in the presence of the B -field or with the Chan-Paton factor, the coefficient of the noncommuting factors contributing to $\mathcal{A}_B^{(4)}$ in the different channels, can be summed up to write a single integral over the real numbers[44]. In [45], which considers introducing the Chan-Paton factor in p -adic strings, it was suggested that this property be abandoned. Following this idea, we will consider different channels separately. Consider the t - u channel first: in the ordinary string theory, this is given by:

$$\mathcal{A}_{B,tu}^{(4)} = \int_0^1 d\xi |\xi|^{k_1 \cdot k_3} |1 - \xi|^{k_2 \cdot k_3} \cos \frac{1}{2} (k_1 \theta k_2 + \epsilon(\xi) k_1 \theta k_3 + \epsilon(1 - \xi) k_2 \theta k_3), \quad (44)$$

where, $\epsilon(\xi)$ is the sign function on \mathbf{R} . We will attempt to write an appropriate p -adic analogue of the above. However, there does not exist a unique prescription for this, a fact that was already realized in [39,45] in the context of the Chan-Paton factors. In order to generalize (44) to the p -adic case, we need to insert a suitable projector to restrict the range of integration. There are several possibilities⁸ to achieve this, as has been listed by the authors of [39]:

$$\begin{aligned}\Xi_1(\xi) &= \frac{1}{4} [1 + \text{sgn}_\tau(\xi)] [1 + \text{sgn}_\tau(1 - \xi)] \equiv H_\tau(\xi) H_\tau(1 - \xi), \\ \Xi_2(\xi) &= \frac{1}{2} [1 + \text{sgn}_\tau(\xi) \text{sgn}_\tau(1 - \xi)], \\ \Xi_3(\xi) &= \frac{1}{2} [\text{sgn}_\tau(\xi) + \text{sgn}_\tau(1 - \xi)],\end{aligned}\tag{45}$$

where, $H_\tau(\xi) = \frac{1}{2} (1 + \text{sgn}_\tau(\xi))$ is the p -adic analogue of the Heaviside function. Inserting one of the step functions above, we can thus define:

$$\begin{aligned}\mathcal{A}_{tu,\Xi}^{(4)} &= \int_{\mathbf{Q}_p} d\xi \Xi(\xi) |\xi|_p^{k_1 \cdot k_3} |1 - \xi|_p^{k_2 \cdot k_3} \\ &\quad \times \cos \frac{1}{2} (\text{sgn}_\tau(-1) k_1 \theta k_2 + \text{sgn}_\tau(-\xi) k_1 \theta k_3 + \text{sgn}_\tau(1 - \xi) k_2 \theta k_3).\end{aligned}\tag{46}$$

In the following, we will study the different choices in turn and examine whether a field theoretic interpretation is possible in each case.

Let us consider Ξ_1 first.

$$\begin{aligned}\mathcal{A}_{tu,\Xi_1}^{(4)} &= \frac{1}{4} \cos \frac{k_1 \theta k_4 + k_2 \theta k_3}{2} \left[\Gamma(\alpha(s)) \Gamma(\alpha(t)) \Gamma(\alpha(u)) - \Gamma_\tau(\alpha(s)) \Gamma_\tau(\alpha(t)) \Gamma(\alpha(u)) \right. \\ &\quad \left. - \Gamma_\tau(\alpha(s)) \Gamma(\alpha(t)) \Gamma_\tau(\alpha(u)) + \Gamma(\alpha(s)) \Gamma_\tau(\alpha(t)) \Gamma_\tau(\alpha(u)) \right] \\ &= \frac{1}{2} \cos \frac{k_1 \theta k_4 + k_2 \theta k_3}{2} \left[\frac{p-3}{2p} + \frac{p-1}{p} \left\{ \frac{1}{p^{\alpha(u)} - 1} + \frac{1}{p^{\alpha(t)} - 1} \right\} \right].\end{aligned}\tag{47}$$

Upon summing over the three, namely, t - u , u - s and s - t channels, we find that the resulting Veneziano amplitude agrees qualitatively with the calculation from the effective field theory (41).

⁸ Our choice, however, is somewhat more restricted by the fact that we consider only $\tau = p$, $p \equiv 3 \pmod{4}$, so as to ensure $\text{sgn}_\tau(-\xi) = -\text{sgn}_\tau(\xi)$. This limits our options compared to [39].

For the other choices of the step function in (46), the results are:

$$\begin{aligned}\mathcal{A}_{tu,\Xi_2}^{(4)} &= c_{12} (c_{13}c_{23} + s_{13}s_{23}) \mathcal{B}_{tu,\Xi_2}^{(4)} - s_{12} (s_{13}c_{23} - c_{13}s_{23}) \mathcal{B}_{tu,\Xi_3}^{(4)} \\ \mathcal{A}_{tu,\Xi_3}^{(4)} &= c_{12} (c_{13}c_{23} + s_{13}s_{23}) \mathcal{B}_{tu,\Xi_3}^{(4)} - s_{12} (s_{13}c_{23} - c_{13}s_{23}) \mathcal{B}_{tu,\Xi_2}^{(4)},\end{aligned}\quad (48)$$

where,

$$\mathcal{B}_{tu,\Xi}^{(4)} = \int_{\mathbf{Q}_p} d\xi \Xi(\xi) |\xi|_p^{k_1 \cdot k_3} |1 - \xi|_p^{k_2 \cdot k_3}. \quad (49)$$

Explicitly,

$$\begin{aligned}\mathcal{B}_{tu,\Xi_2}^{(4)} &= \frac{1}{2} \left[\Gamma(\alpha(s)) \Gamma(\alpha(t)) \Gamma(\alpha(u)) + \Gamma(\alpha(s)) \Gamma_\tau(\alpha(t)) \Gamma_\tau(\alpha(u)) \right] \\ &= \frac{p-1}{2p} \left(1 + \frac{1}{p^{\alpha(t)} - 1} + \frac{1}{p^{\alpha(u)} - 1} \right), \\ \mathcal{B}_{tu,\Xi_3}^{(4)} &= -\frac{1}{2} \left[\Gamma_\tau(\alpha(s)) \Gamma(\alpha(t)) \Gamma_\tau(\alpha(u)) + \Gamma_\tau(\alpha(s)) \Gamma_\tau(\alpha(t)) \Gamma(\alpha(u)) \right] \\ &= -\frac{1}{p} + \frac{p-1}{2p} \left(\frac{1}{p^{\alpha(t)} - 1} + \frac{1}{p^{\alpha(u)} - 1} \right).\end{aligned}\quad (50)$$

Hence:

$$\begin{aligned}\mathcal{A}_{tu,\Xi_{2,3}}^{(4)} &= \cos \frac{1}{2} (k_1 \theta k_4 + k_2 \theta k_3) \left[\frac{p-3}{4p} + \frac{p-1}{2p} \left(\frac{1}{p^{\alpha(t)} - 1} + \frac{1}{p^{\alpha(u)} - 1} \right) \right] \\ &\quad \pm \cos \frac{1}{2} (k_1 \theta k_2 + k_2 \theta k_3 + k_3 \theta k_1) \frac{p+1}{4p},\end{aligned}\quad (51)$$

the two choices differ by the sign of the term in the second line. When we sum over the three channels, the result does not match that obtained from a field theory—it is the second term which is not compatible. Thus, among the step functions in (45), only $\Xi_1(\xi)$ allows for a field-theoretical interpretation. There is one drawback of this prescription, however. Upon turning off the background B -field, the amplitude does not reduce to the original amplitude. The difficulty here is reminiscent of the same problem encountered in introducing Chan-Paton factors. It is likely that the source of the problem and possibly its solution have the same origin.

Formally, this prescription can be extended to higher-point amplitudes in one of the channels by insertion of the Heaviside functions as:

$$\begin{aligned}& \int_{\mathbf{Q}_p} d\xi_3 \cdots d\xi_{N-1} \prod_{I=3}^{N-1} |\xi_I|_p^{k_1 \cdot k_I} |1 - \xi_I|_p^{k_2 \cdot k_I} H_\tau(\xi_I) H_\tau(1 - \xi_I) \\ & \times \prod_{\substack{I,J=3 \\ I < J}}^{N-1} |\xi_I - \xi_J|_p^{k_J \cdot k_J} H_\tau(\xi_J - \xi_I) \exp \left\{ -\frac{i}{2} \sum_{\substack{I,J=1 \\ I < J}}^{N-1} (k_I \theta k_J) \text{sgn}_\tau(\xi_I - \xi_J) \right\}.\end{aligned}\quad (52)$$

Summing over all the permutations under the exchange of momenta of the above amplitude is expected to yield full N -point amplitude. However, we will not attempt to check if this agrees with the corresponding amplitude derived from the effective field theory.

6. Discussions

In summary, we have proposed a coupling for the antisymmetric tensor field B in the p -adic string theory. Specifically, we have added a term corresponding to a constant B -field to the nonlocal action on the boundary \mathbf{Q}_p of the p -adic ‘worldsheet’. The exact Green’s function for the fields $X^\mu(\xi)$ is computed in the presence of the constant B -field. The results parallel the case of the usual string theory in a constant B -field[41,42]: there is a noncommutative factor in the correlation functions of the tachyons and the flat ‘closed string’ metric is replaced by the open string metric $G_{\mu\nu}$. However, the action with the B -field is not invariant under Möbius transformation, a problem that is intimately connected with the fact that there is no natural order among the p -adic numbers. An unfortunate consequence is that the resulting tachyon amplitudes do not quite match those obtained from the noncommutative deformation of the tachyon effective field theory. We also examined the possibility of defining Koba-Nielsen amplitudes so as to derive the field theory results.

Our objective was to derive the noncommutative deformation of the spacetime effective action of the p -adic tachyon, proposed in Ref.[31,32], from a ‘worldsheet’ point of view. In spite of having some encouraging results, this problem remains unresolved. We would expect that for a proper understanding of the B -field, one has to deal with *closed* p -adic strings of which virtually nothing is known. In the early days, in analogy with the usual strings, it was thought that the closed strings have to do with the quadratic extension of \mathbf{Q}_p . Unlike \mathbf{R} , however, neither is the quadratic extension unique, nor is it closed. Indeed, there are an infinite number of finite extensions of \mathbf{Q}_p and none of these is a closed field. Moreover, each finite extension is isomorphic to the boundary of a tree (a Bethe lattice) with appropriate coordination number[10]. Thus, they would correspond to some kind of generalized open strings. It is possible to define a closed field by taking the union of these extensions and augmenting the set by putting in limit points of sequences. The resulting field \mathbf{C}_p , surprisingly, is isomorphic to the set of complex numbers[37], although their topologies are very different. It is likely that closed p -adic strings are to be based on this field \mathbf{C}_p .

Be that as it may, a constant B -field is certainly simpler. First, it only affects the boundary terms. Secondly, one can think of the open p -adic ‘worldsheet’, at least for $p = 2$, as an exotic discretization of the upper half-plane with its Poincare metric[46]. Thus, one should be able to capture the constant flux of the B -field at the vertices of the tree. Finally, let us comment on how one may proceed to couple the B -field to the bulk ‘worldsheet’ action. To this end, we note that the effect of the B -field is topological. We add the pull-back of the 2-form: $X_*(B) = B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\epsilon^{\alpha\beta}$ to the worldsheet action. However, the ‘worldsheet’ of the p -adic string, being a tree, has no closed loop. Hence naively, there is no 2-cycle over which to integrate a 2-form. Nevertheless, we notice that for a prime p (or for any odd integer for that matter), the tree is a bipartite one. We can consistently divide the edges $E = \{e\}$ into two disjoint sets $E_1 = \{e_1\}$ and $E_2 = \{e_2\}$. It is now possible to antisymmetrize derivatives along the edges belonging to these two subsets:

$$B_{\mu\nu} \Delta_{[e_1} X^\mu(z) \Delta_{e_2]} X^\nu(z). \quad (53)$$

Note that this division does not correspond to the sgn_τ function. While the proposal above is unlikely to work as it is, it may be interesting to ponder along these lines.

Note added: A paper [47] (hep-th/0409305) suggesting the same way to couple the B -field in the p -adic string as that considered here, appeared on the arXiv as this manuscript was being finalized. The conclusions there are formal and suffer from subtleties that we have discussed in Sec. 3.

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Appendix A. Material p -adica

In this appendix, we provide a compendium of formulas related to the p -adic number field \mathbf{Q}_p and functions over it. It is not intended as a review, but only to collect in one place most of what is used in this article. For details, we refer the reader to Refs.[35-38].

A.1. p -adic numbers and the Bruhat-Tits tree

An element $\xi \in \mathbf{Q}_p$ may be written as a power series in p :

$$\xi = p^N (\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots) = p^N \sum_{n=0}^{\infty} \xi_n p^n, \quad (\text{A.1})$$

where $N \in \mathbf{Z}$ and $\xi_n \in \{0, 1, \dots, p-1\}$, $\xi_0 \neq 0$. (There is, however, nothing special about this choice—one may work with other representative elements.) Notice the similarity with the Laurent series of meromorphic functions. Addition and multiplication in \mathbf{Q}_p , defined in terms of the series (A.1), give \mathbf{Q}_p the structure of a ring. The subset \mathbf{Z}_p consisting of elements with $N \geq 0$ in (A.1) is a subring known as the p -adic integers. The non-archimedean p -adic norm of ξ is defined to be

$$|\xi|_p = p^{-N}, \quad |0|_p = 0; \quad (\text{A.2})$$

which satisfies the inequality $|\xi - \xi'|_p \leq \max(|\xi|_p, |\xi'|_p)$, which is stronger than the usual triangle inequality. Notice that the norm of a p -adic integer is at most 1.

The series representation (A.1) of a p -adic number also provides us with an isomorphism between \mathbf{Q}_p and the boundary $\partial\mathcal{T}_p$. To see this, let us consider the case $p = 3$ for definiteness and refer to Fig. 1, in which the dashed line denotes the (unique) path through the tree connecting $\xi = \infty$ to $\xi = 0$. Let us label the vertices along this path by C_N , with $N = \pm\infty$ corresponding to zero and infinity respectively and $N = 0$ being the (arbitrarily chosen) point $C = C_0$ at the ‘centre’ of the tree. In order to find a point on the boundary $\partial\mathcal{T}_p$ corresponding to ξ , we start at C_N and then choose the left branch if $\xi_0 = 1$ or the right one if $\xi_0 = 2$. At the next step, we choose one of the branches depending on the value of ξ_1 , and so on. Continuing this way, we arrive at a point on the boundary. This procedure can obviously be generalized for any p .

Coming back to the series (A.1), the finite part $\sum_{n=0}^{N-1} \xi_n p^{n-N}$, consisting of negative powers of p , is called the *fractional part* $\{\xi\}_p$ of ξ . The rest, an infinite series in general, is the *integer part* $[\xi]_p$. Thinking of the fractional part $\{\xi\}_p$ as real, let us define the complex valued function $\chi_p : \mathbf{Q}_p \rightarrow \mathbf{C}$, as

$$\chi_p(\xi) = \exp(2\pi i \{\xi\}_p) = \exp(2\pi i \xi), \quad (\text{A.3})$$

where, in writing the second expression, we have allowed for a little imprecision to regard (A.1) as a formal power series in real. As in the real case, the contribution to (A.3) from the integer part $[\xi]_p$ is trivial and only the fractional part matters. Since $\chi_p(\xi + \xi') = \chi_p(\xi)\chi_p(\xi')$, it is called an *additive character* of \mathbf{Q}_p .

A.2. Fourier transformation

The Fourier transform of a complex valued function⁹ is defined by:

$$\tilde{f}(\omega) = \int_{\mathbf{Q}_p} d\xi \chi_p(-\omega\xi) f(\xi), \quad (\text{A.4})$$

where $d\xi$ is the translationally invariant Haar measure on \mathbf{Q}_p normalized as $\int_{\mathbf{Z}_p} d\xi = 1$. The inverse Fourier transformation is given by

$$f(\xi) = \int_{\mathbf{Q}_p} d\omega \chi_p(\omega\xi) \tilde{f}(\omega). \quad (\text{A.5})$$

This can be proven by use of the formula

$$\int_{B_N} \chi_p(\omega\xi) d\xi = \begin{cases} p^N & \text{for } |\omega|_p \leq p^{-N}, \\ 0 & \text{for } |\omega|_p > p^{-N+1}, \end{cases} \quad (\text{A.6})$$

however, we will omit the proof. It is useful to have the following representation of the delta function $\delta(\xi)$:

$$\delta(\xi) = \int_{\mathbf{Q}_p} d\omega \chi_p(\omega\xi). \quad (\text{A.7})$$

It gives $\int_{\mathbf{Q}_p} d\xi f(\xi) \delta(\xi) = f(0)$ as expected.

A.3. Generalized p -adic gamma and beta functions

The usual p -adic gamma function, called the Gelfand-Graev-Tate gamma function, is defined as:

$$\Gamma(s) = \int_{\mathbf{Q}_p} d\xi \chi_p(\xi) |\xi|_p^{s-1} = \frac{1 - p^{s-1}}{1 - p^{-s}}. \quad (\text{A.8})$$

⁹ Strictly speaking, some of the proofs hold directly for a locally constant function $f(\xi)$, which is a function such that for any $\xi \in \mathbf{Q}_p$, $f(\xi) = f(\xi + \xi')$, whenever $|\xi'|_p \leq p^{l(\xi)}$, for some integer $l(\xi) \in \mathbf{Z}$. The results are then extended to the generalized functions, which are defined as continuous functionals on the linear vector space of locally constant functions.

For any locally constant function $f(\xi)$ with compact support, there exists a maximum value $l(f) = \max_{\xi \in \mathbf{Q}_p} l(\xi)$, called the parameter of constancy. Let \mathcal{D}_m^l be the set of locally constant functions with support contained in $B_m = \{\xi \in \mathbf{Q}_p \mid |\xi|_p \leq p^m\}$ and with the parameter of constancy at least l : $\mathcal{D}_m^l = \{f \in \mathcal{D} \mid \text{supp } f \subset B_m, l(f) \geq l\}$. The Fourier transform $\tilde{f}(\omega)$ of $f(\xi) \in \mathcal{D}_m^l$ turns out to be in \mathcal{D}_{-l}^{-m} .

It has only one singular point at $s = 0$, a simple pole with residue $(p - 1)/p \ln p$. The generalized gamma function associated with sgn_τ is:

$$\begin{aligned}\Gamma_\tau(s) &= \int_{\mathbf{Q}_p} d\xi \chi_p(\xi) |\xi|_p^{s-1} \text{sgn}_\tau(\xi) \\ &= \pm \sqrt{\text{sgn}_\tau(-1)} p^{s-\frac{1}{2}} \quad \text{for } p = p, \varepsilon p.\end{aligned}\tag{A.9}$$

This is an entire function with no singularity on the s -plane. Using the above, we can define the following generalized p -adic Beta functions:

$$\begin{aligned}\int_{\mathbf{Q}_p} d\xi |\xi|_p^{x-1} |1 - \xi|_p^{y-1} &= \Gamma(x) \Gamma(y) \Gamma(1 - x - y) \\ \int_{\mathbf{Q}_p} d\xi \text{sgn}_\tau(\xi) |\xi|_p^{x-1} |1 - \xi|_p^{y-1} &= \text{sgn}_\tau(-1) \Gamma(x) \Gamma_\tau(y) \Gamma_\tau(1 - x - y), \\ \int_{\mathbf{Q}_p} d\xi \text{sgn}_\tau(1 - \xi) |\xi|_p^{x-1} |1 - \xi|_p^{y-1} &= \text{sgn}_\tau(-1) \Gamma_\tau(x) \Gamma(y) \Gamma(1 - x - y), \\ \int_{\mathbf{Q}_p} d\xi \text{sgn}_\tau(\xi) \text{sgn}_\tau(1 - \xi) |\xi|_p^{x-1} |1 - \xi|_p^{y-1} &= \Gamma_\tau(x) \Gamma_\tau(y) \Gamma(1 - x - y).\end{aligned}\tag{A.10}$$

A.4. Fourier transform of derivatives

We will now discuss the Fourier transform of the derivatives of a (complex valued) function on \mathbf{Q}_p or its extension to the interior of the ‘worldsheet’ \mathcal{T}_p . Recall the expressions for the normal and tangential derivatives in Eqs.(14) and (9). More generally, the ℓ -th normal derivative is given by

$$\left(\partial_n^{(p)}\right)^\ell f(\xi) = \int_{\mathbf{Q}_p} d\xi' \frac{f(\xi') - f(\xi)}{|\xi' - \xi|_p^{\ell+1}}.\tag{A.11}$$

After a Fourier transform, we obtain

$$\left(\partial_n^{(p)}\right)^\ell f(\xi) = \int_{\mathbf{Q}_p} d\omega \chi_p(\omega \xi) \left(\widetilde{\partial_n^{(p)\ell} f}\right)(\omega),\tag{A.12}$$

where

$$\left(\widetilde{\partial_n^{(p)\ell} f}\right)(\omega) = \widetilde{f}(\omega) \int_{\mathbf{Q}_p} d\xi \frac{\chi_p(-\omega \xi) - 1}{|\xi|_p^{\ell+1}}.\tag{A.13}$$

The integral above is divergent. Therefore, in order to evaluate it, we need to regularize by an ‘infrared’ cutoff. This is done as follows. Let $S_m = \{\xi \in \mathbf{Q}_p \mid |\xi|_p = p^m\}$, the

subset of p -adic numbers with a fixed norm p^m . Clearly, $S_m \cap S_n = \emptyset$ for $m \neq n$, and $\mathbf{Q}_p = \cup_{m \in \mathbb{Z}} S_m$. We define

$$\begin{aligned}
\int_{\mathbf{Q}_p} \frac{d\xi}{|\xi|_p^{\ell+1}} &= \lim_{N \rightarrow \infty} \sum_{m=-N}^{\infty} \int_{S_m} \frac{d\xi}{|\xi|_p^{\ell+1}} \\
&= \frac{p-1}{p} \lim_{N \rightarrow \infty} \sum_{m=-\infty}^N p^{m\ell}, \\
\int_{\mathbf{Q}_p} d\xi \frac{\chi_p(\omega\xi)}{|\xi|_p^{\ell+1}} &= \lim_{N \rightarrow \infty} \sum_{m=-N}^{\infty} \int_{S_m} d\xi \frac{\chi_p(\omega\xi)}{|\xi|_p^{\ell+1}} \\
&= \lim_{N \rightarrow \infty} \left(\sum_{m=k}^N p^{m\ell} - \sum_{m=k-1}^N p^{m\ell-1} \right) \\
&= \lim_{N \rightarrow \infty} \left[\frac{p-1}{p} \sum_{m=k}^N p^{m\ell} - p^{(k-1)\ell-1} \right],
\end{aligned} \tag{A.14}$$

where $k \in \mathbb{Z}$ is an integer such that $|\omega|_p = p^k$. In the above, we have used the fact that in our normalization $\int_{S_m} d\xi = (p-1)p^{m-1}$. Thus we obtain

$$\int_{\mathbf{Q}_p} d\xi \frac{\chi_p(-\omega\xi) - 1}{|\xi|_p^{\ell+1}} = |\omega|_p^\ell \frac{1 - p^{-\ell-1}}{1 - p^\ell}, \tag{A.15}$$

specifically for $\ell = 1$:

$$\left(\widetilde{\partial_n^{(p)}} f \right) (\omega) = \frac{1 - p^{-2}}{1 - p} |\omega|_p \tilde{f}(\omega). \tag{A.16}$$

Finally let us turn to the Fourier transform of the tangential derivative. After a change of the variable in (A.9), one finds

$$\frac{\text{sgn}_\tau(-\omega)}{|\omega|_p^s} \Gamma_\tau(s) = \int_{\mathbf{Q}_p} d\xi \chi_p(-\omega\xi) |\xi|_p^{s-1} \text{sgn}_\tau(\xi). \tag{A.17}$$

Using the above, it is easy to find the Fourier transform of the tangential derivative:

$$\begin{aligned}
\left(\widetilde{\partial_t^{(p)}} f \right) (\omega) &= \int_{\mathbf{Q}_p} d\xi d\xi' \frac{\text{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} \chi_p(-\omega\xi) f(\xi') \\
&= \Gamma_\tau(-1) \text{sgn}_\tau(-\omega) |\omega|_p \tilde{f}(\omega).
\end{aligned} \tag{A.18}$$

Let us reiterate that $\Gamma_\tau(-1)$ is not singular.

Appendix B. A regularized integral

In this appendix, we provide a brief derivation of an integral that was used in the main part of the text in finding the solution (24) of the ‘equation of motion’, or equivalently, the Green’s function (25). The integral in question is

$$\int_{\mathbf{Q}_p} d\omega \frac{\chi_p(\omega\xi) - \chi_p(\omega\xi')}{|\omega|_p} = -\frac{p-1}{p \ln p} \ln \frac{|\xi|_p}{|\xi'|_p}, \quad (\text{B.1})$$

which follows from the ‘infrared regularized’ expression

$$\int_{\mathbf{Q}_p} d\omega \frac{\chi_p(\omega\xi)}{|\omega|_p} = \frac{p-1}{p \ln p} \left[\lim_{\alpha \rightarrow 0} \left(\frac{1}{\alpha} \right) - \ln |\xi|_p \right]. \quad (\text{B.2})$$

In order to obtain this, we consider the following more general integral, well-defined for $\alpha > 0$, and evaluate it as:

$$\begin{aligned} \int_{\mathbf{Q}_p} |\omega|_p^{\alpha-1} \chi_p(\omega\xi) d\omega &= \sum_{m=-\infty}^{\infty} \int_{S_m} |\omega|_p^{\alpha-1} \chi_p(\omega\xi) d\omega \\ &= \frac{1}{|\omega|_p^\alpha} \left(\frac{1 - p^{\alpha-1}}{1 - p^{-\alpha}} \right), \end{aligned} \quad (\text{B.3})$$

where $k \in \mathbf{Z}$ is an integer such that $|\omega|_p = p^k$. The regularized expression (B.2) is given in the limit $\alpha \rightarrow 0$. The procedure here exactly parallels the case of standard two-dimensional theories, such as the one for the worldsheet theory of the usual strings.

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